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ON THE THEORY OF MESONS

BY

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INTRODUCTION

In order to obtain a consistent theory of nuclear forces by means of intermediary meson fields according to the idea of YUKAWA[1]), it has been necessary to introduce a considerable number of independent field variables for the description of the meson field[2]). As was shown in a recent paper in these Proceedings by L. ROSENFELD and the present author*, the meson field variables are to be taken as a combination of the field quantities of the "vector" theory with those of the "pseudoscalar" theory. These two types of field variables are so far completely independent and satisfy equations which are separately covariant under arbitrary changes of the frame of reference. Each of the two types of field equations contains a set of two universal constants, g_1', g_2', and f_1', f_2', which determine the strength of the couplings between the nucleons** and the "vector" and "pseudoscalar" meson fields respectively. In order to avoid the occurrence of strongly singular terms already in the static interaction energy, the absolute value of g_2' must

* C. MØLLER and L. ROSENFELD, D. Kgl. Danske Vidensk. Selskab, Math.-fys. Medd. XVII, 8 (1940), later referred to as M. R.

** In a recent note on the theory of nuclear forces[9]), the word nuclon originally proposed by BELINFANTE, has been used as a common notation for the heavy nuclear constituents, neutrons and protons. In the meantime, howewer, it has been pointed out to me that, since the root of the word nucleus is "nucle", the notation "nucleon" would from a philological point of view be more appropriate for this purpose.

be chosen equal to the absolute value of f_2' but, otherwise, the constants are completely independent. Although the theory has all the defects inherent in any quantum field theory, it was shown that all processes in which distances larger than a given critical "universal" distance are of importance only, can be treated unambiguously by a "correspondence" method.

Quite apart from the fundamental difficulties of quantum field theory, which probably can only be removed by an appropriate incorporation of the "universal" length into the theory[3]), the occurrence of two independent types of fields and four universal constants in the theory is an unsatisfactory feature which arouses the suspicion that the formalism developed in M.R. is only part of a more comprehensive formalism in which the vector and pseudoscalar meson fields are more intimately connected and, consequently, the number of independent constants is reduced.

In the first section of the present paper, it is shown that the vector and pseudoscalar field equations may be comprised in a five-dimensional representation which is invariant under all rotations in a five-dimensional space, if the constants f_1' and f_2' are fixed by the equations

$$f_1' = g_1', \quad f_2' = -g_2'.$$

These equations are in accordance with the condition $f_2'^2 = g_2'^2$ mentioned above. Since the Lorentz transformations form a subgroup of the group of rotations in five-dimensional space, the invariance properties of the field equations are in this way extended to a wider group of transformations in which the field variables of the "vector" and "pseudoscalar" theories are transformed into each other. In the new formulation of the meson theory, we are, thus, left with only two independent constants which may be fixed empirically, for instance by a comparison of the theoretically and experimentally determined energy levels of the deuteron.

In section 2 of the present paper, the physical interpretation of the formalism developed in section 1 is discussed. Since the group of five-dimensional rotations is equivalent to the group of space-time rotations and translations in de-SITTER space, it is possible to interpret the equations of section 1 as the field equations of the meson theory in de-SITTER space. In the approximation where the curvature of this space is neglected, all consequences of the theory are the same as in the old formulation, apart from the difference arising from the fixation of the constants f'_1 and f'_2 which, of course, makes the predictions of the new theory more precise.

In the last two sections, the equations of motion are written in Hamiltonian form, the expression for the static interaction potential is derived, and the bearing of the theory on the problem of the electric quadrupole moment of the deuteron is briefly discussed. It is shown that the value of this quantity, on the present theory, is of the same order of magnitude as the value derived from the measurements of RABI and his collaborators[4]). An exact determination of the value and sign of the quadrupole moment will, however, require a closer examination*.

* The definite statement on the sign made in a recent note[9]) was premature.

1. Mathematical formalism of the meson theory.

Before proceeding to the main subject of this section, we shall give a brief account of the formalisms of the vector and pseudoscalar theories. In either case, the assumption of both charged and neutral mesons necessitates the introduction of three independent sets of real wave functions F_1, F_2, and F_3, where the index **3** refers to the neutral field, while the two other sets of quantities, F_1 and F_2, together describe the charged mesons. The differential equations of these fields may be compactly written as symbolical vector equations if, following the procedure in M.R., corresponding components of the three sets of field variables are grouped into symbolical vectors denoted by

$$\boldsymbol{F} = (F_1, F_2, F_3).$$

If, similarly, the corresponding source densities S_1, S_2, S_3, giving rise to the real fields in question are considered as the components of a symbolical vector

$$\boldsymbol{S} = (S_1, S_2, S_3)$$

the field equations of the three kinds of fields simply appear as different components of symbolical vector equations. We shall now write down the field equations of the vector and pseudoscalar theory in a form which shows their covariance against Lorentz transformations.

Let $x_4 = ict$ be the usual imaginary time variable, while x_1, x_2, x_3 are the ordinary Cartesian space coordinates. In the vector meson theory, the field is now described by a four-vector $\boldsymbol{U}_k$ and an antisymmetrical tensor $\boldsymbol{G}_{kl} = - \boldsymbol{G}_{lk}$. Similarly, the source densities giving rise to these fields are described by a four-vector $\boldsymbol{M}_k$ and an antisymmetrical tensor $\boldsymbol{S}_{kl} = -\boldsymbol{S}_{lk}$, both of which are functions of the variables of the nucleons alone. The field equations may then be written

$$\left.\begin{aligned} \boldsymbol{G}_{kl} &= \frac{\partial \boldsymbol{U}_l}{\partial x_k} - \frac{\partial \boldsymbol{U}_k}{\partial x_l} + \boldsymbol{S}_{kl} \\ \sum_{l=1}^{4} \frac{\partial \boldsymbol{G}_{kl}}{\partial x_l} &+ \kappa^2 \boldsymbol{U}_k = \boldsymbol{M}_k \quad (k, l) = (1, 2, 3, 4) \end{aligned}\right\} \tag{1}$$

where κ is the constant determining the range of the nuclear forces. In the pseudoscalar meson theory, the field and the sources are described by pseudoscalars $\boldsymbol{\Psi}$ and $\boldsymbol{R}$ and pseudovectors $\boldsymbol{\Gamma}_k$ and $\boldsymbol{P}_k$, the source densities $\boldsymbol{R}$ and $\boldsymbol{P}_k$ being again functions of the variables of the nucleons, and the corresponding field equations are

$$\left.\begin{aligned} \boldsymbol{\Gamma}_k &= -\frac{\partial \boldsymbol{\Psi}}{\partial x_k} + \boldsymbol{P}_k \\ \sum_{k=1}^{4} \frac{\partial \boldsymbol{\Gamma}_k}{\partial x_k} &+ \kappa^2 \boldsymbol{\Psi} = \boldsymbol{R}. \end{aligned}\right\} \tag{1'}$$

In order to express the source quantities as functions of the dynamical variables of the nucleons, it is convenient to introduce five quantities γ_μ ($\mu = 0, 1, 2, 3, 4,$) defined by

$$\left.\begin{aligned} \gamma_k &= -i\beta\alpha_k = \rho_2\sigma_k \quad (k = 1, 2, 3) \\ \gamma_4 &= \beta = \rho_3 \\ \gamma_0 &= \gamma_1\gamma_2\gamma_3\gamma_4 = -\rho_1 \end{aligned}\right\} \tag{2}$$

where β and (α_k, ρ_k, σ_k) ($k = 1, 2, 3$) are the ordinary Dirac matrices.

The variables γ_μ obviously satisfy the commutation rules

$$\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2\,\delta_{\mu\nu} \qquad (\mu, \nu = 0, 1, 2, 3, 4). \tag{3}$$

Let ψ be the wave function of the heavy particles, and

$$\psi^{\dagger} = i\psi^{*}\beta \tag{4}$$

where ψ^{*} is the conjugate complex of ψ.

If, furthermore, $\boldsymbol{\tau} = (\tau_1, \tau_2, \tau_3)$ is the "isotopic spin-vector" chosen in such a way that the eigenvalues $+1$ and -1 of τ_3 refer to the neutron and proton states, respectively, the source densities occurring in (1) and (1′) are simply given by*

$$\left.\begin{aligned} \boldsymbol{M}_k &= g_1 \psi^{\dagger} \boldsymbol{\tau} \gamma_k \psi \quad (k = 1, 2, 3, 4) \\ \boldsymbol{R} &= f_1 \psi^{\dagger} \boldsymbol{\tau} \gamma_0 \psi \end{aligned}\right\} \tag{5}$$

$$\left.\begin{aligned} \boldsymbol{S}_{kl} &= \frac{g_2}{2\varkappa} \psi^{\dagger} \boldsymbol{\tau} [\gamma_k, \gamma_l] \psi \\ \boldsymbol{P}_k &= \frac{f_2}{2\varkappa} \psi^{\dagger} \boldsymbol{\tau} [\gamma_0, \gamma_k] \psi \end{aligned}\right\} \tag{6}$$

$$(k, l) = (1, 2, 3, 4)$$

where $[\gamma_\mu, \gamma_\nu] = \gamma_\mu \gamma_\nu - \gamma_\nu \gamma_\mu$ is the commutator of γ_μ and γ_ν.

The quantities (5) and (6) are real, apart from the four-components which are purely imaginary, and this condition is not altered by Lorentz transformations.

As emphasized by Klein[5]), the quantities occurring in (5) and (6) are covariant for a wider group of transformations than Lorentz transformations. If we formally introduce a new real variable x_0, the five quantities (x_μ) $\mu = 0, 1, 2, 3, 4$ may be interpreted as coordinates of a point in a Euclidian space $\{R_5\}$ of five dimensions. Considering now an orthogonal transformation

* While the constants g_1, f_1, and f_2 are the same as those used in M. R., the constant g_2 differs from the corresponding quantity in M. R. by a factor of -1, i. e. $f_1 = f_1'$, $f_2 = f_2'$, $g_1 = g_1'$, but $g_2 = -g_2'$.

$$x'_\mu = \sum_{\nu=0}^{4} a_{\mu\nu} x_\nu, \quad \sum_{\rho=0}^{4} a_{\mu\rho} a_{\nu\rho} = \sum_{\rho=0}^{4} a_{\rho\mu} a_{\rho\nu} = \delta_{\mu\nu}$$

$$\Delta_5 = |a_{\mu\nu}| = +1$$

in $\{R_5\}$, the quantities

$$\left.\begin{aligned} t_\mu &= \Psi^\dagger \gamma_\mu \Psi \\ m_{\mu\nu} &= -m_{\nu\mu} = \Psi^\dagger [\gamma_\mu, \gamma_\nu] \Psi \\ (\mu, \nu) &= (0, 1, 2, 3, 4) \end{aligned}\right\} \tag{8}$$

will transform like the components of a five-vector and of an antisymmetrical tensor in $\{R_5\}$.

For instance, the transformed variables t'_μ will be connected with the old variables by the equation

$$t'_\mu = \sum_{\nu=0}^{4} a_{\mu\nu} t_\nu$$

and the transformed wave function Ψ' will be connected with Ψ by the relation

$$\Psi' = S\Psi$$

S being a unitary operator which, for an infinitesimal transformation

$$x'_\mu = x_\mu + \sum_{\mu=0}^{4} \varepsilon_{\mu\nu} x_\nu, \quad \varepsilon_{\mu\nu} = -\varepsilon_{\nu\mu} \tag{9}$$

is given by

$$S = 1 + \frac{1}{8} \sum_{\mu,\nu=0}^{4} \varepsilon_{\mu\nu} [\gamma_\mu, \gamma_\nu]. \tag{9'}$$

On account of (2) and (3) we have, therefore, $S^* = \beta S^{-1} \beta$, where S^* is the Hermitian conjugate of S, and thus for the transformed variable $\Psi'^\dagger$ we get

$$\Psi'^\dagger = \Psi^\dagger S^{-1}.$$

This important relation only holds for such transformations which preserve the reality of the variables x_0, x_1, x_2, x_3, and the imaginary character of x_4. The determinant of any transformation defined by (9) and (9′) is positive.

It is tempting now to comprise the five quantities (5) in one quantity $(\boldsymbol{M}_\mu)$ with five components defined by

$$\boldsymbol{M}_\mu = \begin{cases} \boldsymbol{R} & \mu = 0 \\ \boldsymbol{M}_k & \mu = k = 1, 2, 3, 4. \end{cases} \tag{10}$$

According to (8), $(\boldsymbol{M}_\mu)$ will be a five-vector if $f_1 = g_1$. Similarly, the quantity defined by

$$\boldsymbol{S}_{\mu\nu} = -\boldsymbol{S}_{\nu\mu} = \begin{cases} \boldsymbol{P}_k & \mu = 0, \nu = k = 1, 2, 3, 4 \\ \boldsymbol{S}_{kl} & \left.\begin{matrix} \mu = k \\ \nu = l \end{matrix}\right\} = (1, 2, 3, 4) \end{cases} \tag{10'}$$

will be a tensor in $\{R_5\}$ if $f_2 = g_2$.

Putting now

$$\boldsymbol{U}_\mu = \left.\begin{cases} \boldsymbol{\Psi} & \mu = 0 \\ \boldsymbol{U}_k & \mu = k = 1, 2, 3, 4 \end{cases}\right\} \tag{11}$$

and

$$\boldsymbol{G}_{\mu\nu} = -\boldsymbol{G}_{\nu\mu} = \left.\begin{cases} \boldsymbol{\Gamma}_k & \mu = 0, \nu = k = 1, 2, 3, 4 \\ \boldsymbol{G}_{kl} & \left.\begin{matrix} \mu = k \\ \nu = l \end{matrix}\right\} = 1, 2, 3, 4 \end{cases}\right\} \tag{11'}$$

the equations (1) and (1′) may be compactly written

$$\left.\begin{aligned} \boldsymbol{G}_{\mu\nu} &= \frac{\partial \boldsymbol{U}_\nu}{\partial x_\mu} - \frac{\partial \boldsymbol{U}_\mu}{\partial x_\nu} + \boldsymbol{S}_{\mu\nu} \\ \frac{\partial \boldsymbol{G}_{\mu\nu}}{\partial x_\nu} &+ \kappa^2 \boldsymbol{U}_\mu = \boldsymbol{M}_\mu \\ (\mu, \nu) &= (0, 1, 2, 3, 4) \end{aligned}\right\} \tag{12}$$

if we assume that the field variables in (1) and (1′) do not depend on the coordinate x_0. Here we have made the convention that Greek indices which appear twice, like ν in the last equation (12), have to be summed over all values from 0 to 4.

The field equations are, thus, brought into a form in which their invariance against all rotations in $\{R_5\}$ is apparent, provided the field variables and source densities in (12) transform like tensors. As mentioned above, the source quantities will be tensors if (and only if)

$$f_1 = g_1 \qquad f_2 = g_2 \tag{13}$$

and the field variables $\boldsymbol{G}_{\mu\nu}$ and $\boldsymbol{U}_\mu$ may by definition be taken to be tensors.

To obtain a real covariance of (12) of the kind mentioned above we must, of course, admit that the field variables and source densities in general may be functions of all "coordinates" (x_μ) including x_0. We are, thus, led to the view that the meson theory developed in M.R. represents only a special case of a more general theory in which the field variables are characterized by five parameters (x_μ) $\mu = 0, 1, 2, 3, 4$, instead of the ordinary four space and time coordinates. A physical interpretation of this formalism and especially of the variables x_μ will be given in the next section, and we shall here let the question open whether the values of (x_μ) are confined to a finite domain ω in $\{R_5\}$ or cover the whole five-dimensional space. We only remark that the region ω, if finite, must have a form which is invariant under all rotations in $\{R_5\}$, i. e. the boundary of ω must consist of concentric "spheres".

The general field equations are, then, given by (12) with

$$\left.\begin{aligned} \boldsymbol{M}_\mu &= g_1 \Psi^\dagger \boldsymbol{\tau} \gamma_\mu \Psi \\ \boldsymbol{S}_{\mu\nu} &= \frac{g_2}{2\kappa} \Psi^\dagger \boldsymbol{\tau} [\gamma_\mu, \gamma_\nu] \Psi. \end{aligned}\right\} \tag{14}$$

To be consistent, we have now also to generalize the equations of motion of the nucleons to make them invariant against all rotations in $\{R_5\}$. This is easily done, and we get for the wave equation of these particles

$$\left\{\gamma_\mu \frac{\partial}{\partial x_\mu} + \frac{M_0 c}{\hbar} - \frac{i g_1}{\hbar c} \boldsymbol{U}_\mu \boldsymbol{\tau} \gamma_\mu + \frac{i g_2}{4 \varkappa \hbar c} \boldsymbol{G}_{\mu\nu} \boldsymbol{\tau} [\gamma_\mu, \gamma_\nu]\right\} \psi = 0 \quad (15)$$

and the adjoint equation

$$\psi^\dagger \left\{\gamma_\mu \frac{\partial}{\partial x_\mu} - \frac{M_0 c}{\hbar} + \frac{i g_1}{\hbar c} \boldsymbol{U}_\mu \boldsymbol{\tau} \gamma_\mu - \frac{i g_2}{4 \varkappa \hbar c} \boldsymbol{G}_{\mu\nu} \boldsymbol{\tau} [\gamma_\mu, \gamma_\nu]\right\} = 0 \quad (15')$$

the differentiation in the last equation operating backwards on the function $\psi^\dagger$. Here, M_0 is an abbreviation for

$$M_0 = \frac{1+\tau_3}{2} \overset{0}{M}_N + \frac{1-\tau_3}{2} \overset{0}{M}_P$$

where $\overset{0}{M}_N$ and $\overset{0}{M}_P$ are the ordinary rest masses of neutron and proton. The equations (15) and (15′) are obviously covariant under all rotations in $\{R_5\}$ and, if ψ does not depend upon x_0, they become identical with the wave equations of the heavy particles following from the vector and pseudoscalar meson theory.

Multiplying (15) by $\psi^\dagger$ on the left, and (15′) by ψ on the right, and adding we get

$$\frac{\partial}{\partial x_\mu} \psi^\dagger \gamma_\mu \psi = 0. \quad (16)$$

Multiplying (15) by $\frac{1}{2} \psi^\dagger \tau_3$ on the left and (15′) by $\frac{1}{2} \tau_3 \psi$ on the right and adding gives*

$$-\frac{\partial}{\partial x_\mu} \frac{1}{2} \psi^\dagger \tau_3 \gamma_\mu \psi = \frac{1}{\hbar c}\left[\left(\boldsymbol{U}_\mu \wedge \boldsymbol{M}_\mu\right)_3 - \frac{1}{2}\left(\boldsymbol{G}_{\mu\nu} \wedge \boldsymbol{S}_{\mu\nu}\right)_3\right] \quad (17)$$

* The symbol $\wedge$ indicates a symbolical vector product, i. e. $(\boldsymbol{A} \wedge \boldsymbol{B})_3 = A_1 B_2 - A_2 B_1$.

In this derivation we have used the relations

$$[\tau_3, \tau_1] = 2\,i\,\tau_2 \text{ and } [\tau_3, \tau_2] = -2\,i\,\tau_1$$

characteristic for any spin vector. Using the field equations (12), we get further

$$\frac{\partial}{\partial x_\mu}\frac{1}{\hbar c}\left(\boldsymbol{G}_{\mu\nu} \wedge \boldsymbol{U}_\nu\right)_3 = \frac{1}{\hbar c}\left[\left(\boldsymbol{U}_\mu \wedge \boldsymbol{M}_\mu\right)_3 - \frac{1}{2}\left(\boldsymbol{G}_{\mu\nu} \wedge \boldsymbol{S}_{\mu\nu}\right)_3\right]. \quad (18)$$

From (16), (17) and (18) it follows at once that the five-vector

$$s_\mu = \psi^\dagger \frac{1-\tau_3}{2}\gamma_\mu \psi - \frac{1}{\hbar c}\left(\boldsymbol{G}_{\mu\nu} \wedge \boldsymbol{U}_\nu\right)_3 \quad (19)$$

satisfies the divergence relation

$$\frac{\partial s_\mu}{\partial x_\mu} = 0 \quad (20)$$

The vector $\psi^\dagger \gamma_\mu \psi$ in (16) and the vector s_μ may be interpreted as the particle-current and charge-current density vectors in $\{R_5\}$.

The equations of motion (12), (15) and (15′) may be derived from the variational principle

$$\delta \overline{L} = 0 \quad (21)$$

with the total Lagrangeian

$$\left.\begin{aligned} \overline{L} = \int_\omega \Big\{ & -\frac{1}{4}\boldsymbol{G}_{\mu\nu}\boldsymbol{G}_{\mu\nu} - \frac{\kappa^2}{2}\boldsymbol{U}_\mu \boldsymbol{U}_\mu + \boldsymbol{U}_\mu \boldsymbol{M}_\mu \\ & - \frac{\hbar c}{i}\psi^\dagger\left[\gamma_\mu \frac{\partial}{\partial x_\mu} + \frac{M_0 c}{\hbar}\right]\psi \Big\}\, dx_0\, dx_1\, dx_2\, dx_3\, dx_4. \end{aligned}\right\} \quad (22)$$

Here, ω denotes the region in $\{R_5\}$ covered by the possible (x_μ)-values, $\boldsymbol{G}_{\mu\nu}$ is defined by the first equations (12), and ψ, $\psi^\dagger$ and $\boldsymbol{U}_\mu$ are supposed to be varied independently in such a way that the variation of these quantities is zero everywhere on the boundary of the region ω. The corresponding Euler equations are then identical with the equations of motion (12), (15) and (15′), if the wave functions ψ and $\psi^\dagger$ provisorily are treated as c-numbers.

2. Physical interpretation of the formalism.

We shall begin this section with a brief survey of the properties of the de-Sitter world which, as we shall see, is of importance for the discussion of the physical interpretation of the formalism developed in section 1. As emphasized by Dirac[6]), the de-Sitter space (with no local gravitational fields) represents the only solution of the equations of general relativity which has a non-trivial group of operations carrying the space over into itself. The group of operations in question is just the group of orthogonal transformations in a five-dimensional space considered in section 1. This follows at once from the remark made by Robertson[7]) that the de-Sitter space can be interpreted as the surface of a four-dimensional "sphere" (of hyperbolic character in one direction) embedded in a five-dimensional space $\{R_5\}$. It may, thus, be described by five coordinates x_0, x_1, x_2, x_3, x_4 connected by the relation

$$x_\mu x_\mu = R^2 \tag{23}$$

where four of the coordinates are real and one, x_4 say, is purely imaginary. The radius of the sphere R is connected with Einstein's cosmological constant λ by the equation

$R = \sqrt{\frac{3}{\lambda}}$ and is of the order of magnitude of 10^{27} cm. Any rotation in $\{R_5\}$ represented by

$$x_\mu' = a_{\mu\nu}\, x_\nu, \quad a_{\mu\rho}\, a_{\nu\rho} = a_{\rho\mu}\, a_{\rho\nu} = \delta_{\mu\nu} \qquad (24)$$
$$\Delta_5 = |\, a_{\mu\nu}\,| = +1$$

obviously leaves (23) invariant.

Further, it may be shown* that the group of rotations (24) induces a group of transformations in the de-Sitter world which is closely analogous to the group of Lorentz transformations (including spatial reflections) and translations in the Minkowski world of special relativity. An arbitrary point P on the sphere (23) corresponds to a certain "event" in the de-Sitter world. In a small four-dimensional region on the sphere around the point P, we may introduce the ordinary space and time coordinates x, y, z, t of special relativity. To see the connection between these variables and the coordinates in $\{R_5\}$, we introduce that system of coordinates (x_μ) in which the point P has the coordinates $(R, 0, 0, 0, 0)$. For the small region in question, we have then simply

$$(x_\mu) = (R, x, y, z, ict) \qquad (25)$$

neglecting terms of higher order in $\frac{x_k}{R}$ $(k = 1, 2, 3, 4)$, and infinitesimal rotations (24) in $\{R_5\}$ will correspond to ordinary Lorentz rotations and translations in space and time inside the small region.

A vector in the de-Sitter space will be a quantity having five components A_μ which by rotations transform like the coordinates, i. e.

$$A_\mu' = a_{\mu\nu}\, A_\nu.$$

* See the appendix and Robertson, loc. cit.[7]).

If (A_μ) represents a vector which has a correspondence in classical physics like the electromagnetic potentials or the ordinary electric charge-current density, its direction will lie in the de-Sitter space and, thus, we have

$$x_\mu A_\mu = 0. \tag{26}$$

In the system of coordinates (25) for a small region at the point P, this condition reduces to $A_0 = 0$, and the four components A_1, A_2, A_3, A_4 may be identified with the ordinary space-time components of the vector in special relativity.

Any physical law in the de-Sitter space must be invariant against all rotations (24) and so be expressible as tensor or spinor equations in $\{R_5\}$. Since all equations in section 1 are of that type, they may naturally be regarded as the equations of meson theory in the de-Sitter space*. But here we meet with the difficulty that the differentiation processes in (12), (15) and (15′) are going outside the de-Sitter sphere and a priori the field functions are only defined on this sphere which represents the physical space. In his paper quoted above[6]), Dirac avoided the corresponding difficulty in the formulation of the electromagnetic theory in de-Sitter space by assuming all field functions to be homogeneous functions of x_μ of some definite degree. In our case, it seems more natural to use the following procedure. We extend the definition of physical space to a five-dimensional region ω between two "spheres" represented by the equations

* This is one possible interpretation of the formalism in section 1. An alternative interpretation may perhaps be provided by the „projective" point of view according to which the variables (x_μ) are regarded as homogeneous coordinates of a four-dimensional projective space. (See, for instance, W. Pauli, Ann. d. Phys. **18**, 306, 1933). Such interpretation would also establish a connection with Kleins unified theory of gravitation and electricity. (O. Klein, Zs. f. Phys. **46**, 188 (1927).)

$$\left.\begin{aligned} \rho^2 = x_\mu x_\mu = \left(R - \frac{d}{2}\right)^2 \\ \rho^2 = x_\mu x_\mu = \left(R + \frac{d}{2}\right)^2 \end{aligned}\right\} \qquad (27)$$

where d is a small but finite length, and the homogeneity condition for the field quantities is replaced by certain conditions at the boundary of this space. This procedure is natural for two reasons: 1° the field vector $\boldsymbol{U}_\mu$ describing the meson field does not satisfy a condition of the type (26) but has a finite component in a direction perpendicular to the de-Sitter space, i. e. the direction of increasing ρ; 2° although the variable ρ itself has no classical analogue, the canonically conjugate momentum to ρ is, as we shall see, intimately connected with a familiar classical quantity, viz. the rest mass of the particles associated with the field considered. For decreasing d the space defined by (27) goes over into the ordinary de-Sitter space, but also for finite d the theory has a simple physical interpretation.

The boundary conditions for the field quantities are obtained by expressing that the total electric charge in the world has to be constant. If P_1 and P_2 are two points on the same arbitrary five-dimensional radius vector and with $\rho = R - \frac{d}{2}$ and $\rho = R + \frac{d}{2}$, respectively, we must have

$$\left(s_\mu \frac{x_\mu}{\rho}\right)\rho^4 \Big|_{P_1} = \left(s_\mu \frac{x_\mu}{\rho}\right)\rho^4 \Big|_{P_2}$$

where s_μ is given by (19). Any field quantity like ψ, $\psi^\dagger$, $\boldsymbol{U}_\nu$ or $\boldsymbol{G}_{\mu\nu}$, therefore, must satisfy boundary conditions of the form

$$\Psi(P_1)\left(R-\frac{d}{2}\right)^2 = \Psi(P_2)\left(R+\frac{d}{2}\right)^2 \tag{28}$$

which are invariant against all rotations in $\{R_5\}$.

Only such solutions of (12), (15) and (15′) which satisfy the boundary conditions (28) can have any physical meaning. Solutions which do not satisfy (28) would represent a state in which the total charge is not conserved.

Let us now consider a certain region around the point $(R, 0, 0, 0, 0)$ with an extension l small compared with R. Since $R \backsim 10^{27}$ cm this region may be very large (practically infinite) compared with the dimensions in atomic physics. In the following, we shall often neglect all terms of the order $\frac{l}{R}$ since they will be of no importance. This procedure is effectively the same as going to the limit $R \to \infty$. In this limit, the physical space ω defined by (27) becomes an infinite plane parallel region of thickness d. Introducing a new coordinate $\xi = x_0 - R$ instead of x_0, this region is defined by

$$-\frac{d}{2} \leq \xi \leq \frac{d}{2}, \quad -\infty < x_k < \infty, \quad (k = 1, 2, 3, 4) \tag{25'}$$

where the four variables x_k are the usual space and time variables of special relativity. A rotation in $\{R_5\}$ with constant x_0 then corresponds to an ordinary Lorentz transformation while an infinitesimal rotation in the $(x_0 x_k)$-plane corresponds to a parallel displacement along the x_k-axis. In this approximation, the boundary conditions (28) simply become

$$\Psi\left(-\frac{d}{2}, x_k\right) = \Psi\left(\frac{d}{2}, x_k\right) \tag{28'}$$

where x_k stands for the variables x_1, x_2, x_3, x_4.

Integration of (20) over all values of ξ shows, by use of (28′), that the four-dimensional divergence of the four-vector

$$j_k = \frac{e}{d}\int_{-\frac{d}{2}}^{\frac{d}{2}}\left\{\Psi^\dagger \frac{1-\tau_3}{2}\gamma_k \Psi - \frac{1}{\hbar c}\left(\boldsymbol{G}_{k\nu} \wedge \boldsymbol{U}_\nu\right)_3\right\} d\xi \qquad (29)$$

$(k = 1, 2, 3, 4)$

vanishes, i. e.

$$\sum_{k=1}^{4} \frac{\partial j_k}{\partial x_k} = 0. \qquad (29')$$

(29), which is the mean value of s_k over all values of ξ multiplied by e, may thus be interpreted as the charge-current density vector in ordinary space-time. If the wave functions do not depend on ξ, (29) becomes identical with the charge-current density vector of the theory developed in M.R.

Consider a small three-dimensional volume in space-time defined by three infinitesimal four-vectors a_k, b_k, c_k. The charge intersecting this volume will then be

$$\begin{vmatrix} a_1 & b_1 & c_1 & j_1 \\ a_2 & b_2 & c_2 & j_2 \\ a_3 & b_3 & c_3 & j_3 \\ a_4 & b_4 & c_4 & j_4 \end{vmatrix} = \frac{e}{d}\int_{-\frac{d}{2}}^{\frac{d}{2}} \begin{vmatrix} a_1 & b_1 & c_1 & s_1 \\ a_2 & b_2 & c_2 & s_2 \\ a_3 & b_3 & c_3 & s_3 \\ a_4 & b_4 & c_4 & s_4 \end{vmatrix} d\xi \qquad (30)$$

which is invariant against all Lorentz rotations and translations. This quantity can easily be expressed in a way which shows its invariance against all rotations in $\{R_5\}$ in the case of finite R. The three vectors a, b and c lying on the de-Sitter sphere now have five components a_μ, b_μ, c_μ satisfying the equations

$$a_\mu x_\mu = b_\mu x_\mu = c_\mu x_\mu = 0.$$

To these vectors corresponds a certain four-dimensional volume containing points with a radius vector between ρ and $\rho + d\rho$. This volume may be represented by a five-vector dv_μ with components of the type[6])

$$dv_0 = \begin{vmatrix} \frac{x_1}{\rho} & a_1 & b_1 & c_1 \\ \frac{x_2}{\rho} & a_2 & b_2 & c_2 \\ \frac{x_3}{\rho} & a_3 & b_3 & c_3 \\ \frac{x_4}{\rho} & a_4 & b_4 & c_4 \end{vmatrix} \frac{\rho^3}{R^3} d\rho$$

and the total charge intersecting the volume (a, b, c) will thus be

$$\frac{e}{d}\int_{R-\frac{d}{2}}^{R+\frac{d}{2}} s_\mu \, dv_\mu = \frac{c}{d}\int_{R-\frac{d}{2}}^{R+\frac{d}{2}} \begin{vmatrix} x_0 & a_0 & b_0 & c_0 & s_0 \\ x_1 & a_1 & b_1 & c_1 & s_1 \\ x_2 & a_2 & b_2 & c_2 & s_2 \\ x_3 & a_3 & b_3 & c_3 & s_3 \\ x_4 & a_4 & b_4 & c_4 & s_4 \end{vmatrix} \frac{\rho^2 \, d\rho}{R^3}. \qquad (30')$$

(30') is obviously invariant against all rotations in $\{R_5\}$ and becomes identical with (30) in the limit of very large R.

Let us now consider the wave equation (15) of the heavy particles neglecting the interaction with the meson fields. Putting

$$p_\mu = \frac{\hbar}{i}\frac{\partial}{\partial x_\mu} \qquad (\mu = 0, 1, 2, 3, 4)$$

this equation may be written*

$$\left\{\gamma_\mu p_\mu + \frac{M_0 c}{i}\right\}\Psi = 0. \qquad (31)$$

The five-vector p_μ will now, in general, not satisfy equations of the type (26). Besides the four components in the direction of the de-Sitter space, which may be interpreted as components of the ordinary energy-momentum vector, (p_μ) will ordinarily have a component in the direction of the "radius vector" ρ. To see the physical meaning of this component,

* In his paper cited above, Dirac has set up a wave equation for electrons in de-Sitter space which in form differs considerably from equation (31). In the limit of large R and small d the consequences of the two equations are, however, practically identical.

we may for a region small compared with R introduce the coordinates ξ and x_k defined by (25′). p_k $(k = 1, 2, 3, 4)$ will then be the ordinary energy-momentum vector and $p_0 = \frac{\hbar}{i} \frac{\partial}{\partial \xi}$ represents the component in the direction ρ.

Multiplication of (31) by the operator

$$\gamma_\mu p_\mu - \frac{M_0 c}{i}$$

gives, on account of (3),

$$\left\{ \sum_{k=1}^{4} p_k^2 + p_0^2 + M_0^2 c^2 \right\} \Psi = 0$$

which shows that a plane wave of the form

$$\Psi = a\, e^{\frac{i}{\hbar}\left(- Et + \vec{p}\vec{x} + p_0 \xi \right)} \tag{32}$$

represents a state in which the particle has the momentum $\vec{p}$, the energy E, and a rest mass

$$M = M_0 \sqrt{1 + \left(\frac{p_0}{M_0 c} \right)^2}. \tag{33}$$

Since ψ in (32) has to satisfy the boundary conditions (28′), the variable p_0 can have only discrete values given by

$$p_0 = n \frac{h}{d} \tag{34}$$

n being an integer.

The possible values of the rest mass are, thus,

$$M = M_0 \sqrt{1 + n^2 \left(\frac{h}{M_0 c d} \right)^2}. \tag{33'}$$

In the limit of very small d, the distance between the possible M-values will be so large that practically the lowest value M_0 is attainable, only.

Similarly, if we consider the field equations of free mesons, we get from (12) by cancelling the source quantities

$$\boldsymbol{G}_{\mu\nu} = \frac{\partial \boldsymbol{U}_\nu}{\partial x_\mu} - \frac{\partial \boldsymbol{U}_\mu}{\partial x_\nu}$$

$$\frac{\partial \boldsymbol{G}_{\mu\nu}}{\partial x_\nu} + \varkappa^2 \boldsymbol{U}_\mu = 0$$

and from these equations

$$\frac{\partial \boldsymbol{U}_\nu}{\partial x_\nu} = 0 \quad \text{and} \quad \frac{\partial^2 \boldsymbol{U}_\mu}{\partial x_\nu \partial x_\nu} - \varkappa^2 \boldsymbol{U}_\mu = 0$$

or

$$\left\{\Delta + \frac{\partial^2}{\partial \xi^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \varkappa^2\right\} \boldsymbol{U}_\mu = 0.$$

The solutions of this equation, satisfying the boundary conditions (28′), are of the form

$$\boldsymbol{U}_\mu = \boldsymbol{a}_\mu \cos\left(\frac{E}{\hbar} t - \frac{\vec{p}\vec{x} + p_0 \xi}{\hbar} + \delta\right)$$

where $E, \vec{p}$ and p_0 are constants connected by the equation

$$\frac{E^2}{c^2} = |p|^2 + p_0^{\,2} + \varkappa^2 \hbar^2 \tag{35}$$

and p_0 again is given by (34).

The equation (35) represents the energy-momentum relation for a particle with rest mass

$$M_m = \frac{\kappa\hbar}{c}\sqrt{1+\left(\frac{p_0}{\hbar\kappa}\right)^2}.$$

Thus, the mesons may exist with different rest masses given by

$$M_m = M_m^0\sqrt{1+n^2\left(\frac{h}{M_m^0 cd}\right)^2} \tag{36}$$

where the minimum value $M_m^0 = \frac{\kappa\hbar}{c}$ is the mass of the meson in YUKAWA's theory. While the variable ξ itself has no classical analogue it is seen that the corresponding "momentum" p_0 is closely connected with the rest mass. It would, therefore, seem more natural to introduce the variable p_0 instead of ξ or x_0 into the field equations. For instance, if we expand the wave function ψ of the heavy particles into a FOURIER series

$$\Psi(x_k, \xi) = \sum_{p_0} \Psi_{p_0}(x_k)\, e^{\frac{i}{\hbar}p_0\xi} \tag{37}$$

where the summation is extended over all values (34) of p_0, the boundary conditions (28′) will be satisfied for arbitrary functions $\psi_{p_0}(x_k)$ of the four variables x_1, x_2, x_3, x_4. Neglecting again the interaction with the meson fields, the differential equation for $\psi_{p_0}(x_k)$ has the form

$$\left\{\sum_{k=1}^{4}\frac{\hbar}{i}\gamma_k\frac{\partial}{\partial x_k}+\gamma_0 p_0+\frac{M_0 c}{i}\right\}\Psi_{p_0}(x_k) = 0 \tag{38}$$

which, by a simple canonical transformation, may be brought into the form of an ordinary DIRAC equation with a rest mass given by (33). If the meson field quantities are also written as Fourier series of the type (37) with Fourier coefficients $\boldsymbol{G}_{\mu\nu}^{p_0}$ and $\boldsymbol{U}_{\nu}^{p_0}$, we get for the charge- and current-density (29)

$$j_k = \sum_{p_0} e \left\{ \psi_{p_0}^{\dagger} \frac{1-\tau_3}{2} \gamma_k \psi_{p_0} - \frac{1}{\hbar c} \left(\boldsymbol{G}_{k\nu}^{p_0} \wedge \boldsymbol{U}_{\nu}^{p_0} \right)_3 \right\} \quad (39)$$

showing that the total charge- and current density is the sum of the corresponding quantities for particles with different values of the rest mass. Each of the terms in this sum has the same form as the charge- and current densities of the theory in M.R. The "Fourier coefficients" ψ_{p_0}, $\boldsymbol{U}_{\nu}^{p_0}$ etc. may, thus, be interpreted as wave functions of the particles with rest masses (33) and (36). Since, however, the use of the variable p_0 instead of ξ or x_0 spoils some of the symmetry in the field equations we shall, in the following, use the original form of these equations given in section 1.

3. Hamiltonian form and quantization of the equations of motion.

Following the general method developed by Belinfante and Rosenfeld[8]), it is easy to construct the "energy-momentum" tensor and the Hamiltonian corresponding to the Lagrangeian (22) of a system of heavy particles and meson fields.

Putting now

$$\left. \begin{array}{ll} i\,\boldsymbol{G}_{r4} = \boldsymbol{F}_r,\ i\,\boldsymbol{S}_{r4} = \boldsymbol{T}_r & (r = 0, 1, 2, 3) \\ \boldsymbol{U}_4 = i\,\boldsymbol{V},\ \boldsymbol{M}_4 = i\,\boldsymbol{N} & \end{array} \right\} \quad (40)$$

the equations (12) may be written*

*) The notation $\dot{A}$ represents the time derivative of A divided by the velocity of light.

$$\left.\begin{aligned}\dot{\boldsymbol{U}}_r &= -\boldsymbol{F}_r - \frac{\partial \boldsymbol{V}}{\partial x_r} + \boldsymbol{T}_r \\ \dot{\boldsymbol{F}}_r &= \kappa^2 \boldsymbol{U}_r + \frac{\partial \boldsymbol{G}_{rs}}{\partial x_s} - \boldsymbol{M}_r\end{aligned}\right\} \quad (41)$$

$$\left.\begin{aligned}\kappa^2 \boldsymbol{V} &= -\frac{\partial \boldsymbol{F}_s}{\partial x_s} + \boldsymbol{N} \\ \boldsymbol{G}_{rs} &= \frac{\partial \boldsymbol{U}_s}{\partial x_r} - \frac{\partial \boldsymbol{U}_r}{\partial x_s} + \boldsymbol{S}_{rs}\end{aligned}\right\} \quad (42)$$

where indices like r, s ... are running from 0 to 3.

We shall, in the following, use a representation of the nuclear particles in configuration space, the i'th particle being then described by the variables

$$q^{(i)} = \left(\tau^{(i)}, \rho^{(i)}, \vec{\sigma}^{(i)}, \vec{x}^{(i)}, \xi^{(i)}\right).$$

Since the variable $x_0 = R + \xi$ will not occur any more we shall for the sake of greater symmetry in the formulae often write x_0 instead of ξ. In the approximation, where R is treated as very large compared with the dimensions of atomic physics, the variables $x_0^{(i)} = \xi^{(i)}$ are confined to the intervals

$$-\frac{d}{2} \leq x_0^{(i)} \leq \frac{d}{2}$$

while the quantities $\left(x_1^{(i)}, x_2^{(i)}, x_3^{(i)}\right) = \vec{x}^{(i)}$ may have practically all values from $-\infty$ to $+\infty$.

If we simply extend the ordinary quantization rules to this configuration space, we get

$$\left.\begin{aligned}&\left[p_r^{(i)}, x_s^{(k)}\right] = \frac{\hbar}{i}\delta^{(ik)}\delta_{rs} \\ &(r, s) = (0, 1, 2, 3)\end{aligned}\right\} \quad (43)$$

where $p_r^{(i)}$ $(r = 1, 2, 3)$ are the components of the ordinary momentum operator of the i'th particle and $p_0^{(i)}$ is the operator connected with the rest mass of the particle. Similarly, the meson field variables $\boldsymbol{F}_r$ and $\boldsymbol{U}_r$ $(r = 0, 1, 2, 3)$ are canonically conjugate variables satisfying the commutation rules

$$\left[F_{\boldsymbol{m},r}(x, t), U_{\boldsymbol{n},s}(x', t)\right] = i\hbar c d\, \delta_{\boldsymbol{mn}}\, \delta_{rs}\, \delta(x_0 - \dot{x}_0')\, \delta(\vec{x} - \vec{x}') \quad (44)$$

all other pairs of variables commuting.

Using (43) and (44), the field equations (41) as well as the equations of motion of the heavy particles may be derived from the usual relation

$$\dot{A} = \frac{i}{\hbar c}[\mathscr{H}, A]$$

with the Hamiltonian

$$\mathscr{H} = \frac{1}{2d}\int \left\{ \boldsymbol{F}_r^2 + \kappa^2 \boldsymbol{V}^2 + \frac{1}{2}\boldsymbol{G}_{rs}^2 + \kappa^2 \boldsymbol{U}_r^2 \right\} d\omega - \frac{1}{8d}\int \boldsymbol{S}_{\mu\nu}^2\, d\omega$$
$$- \frac{1}{d}\int \{\boldsymbol{U}_r \boldsymbol{M}_r + \boldsymbol{F}_r \boldsymbol{T}_r\}\, d\omega \quad (45)$$
$$+ \sum_i \left\{ c\, \alpha_r^{(i)} p_r^{(i)} + \rho_3^{(i)} M_0^{(i)} c^2 \right\}$$

where $$\int d\omega = \int_{-\frac{d}{2}}^{\frac{d}{2}} dx_0 \int dv$$

and the indices r and s are running from 0 to 3. In (45), $\boldsymbol{V}$ and $\boldsymbol{G}_{rs}$ are to be regarded as functions of the dynamical variables defined by the equations (42) and the four variables $\alpha_r = i\beta\gamma_r$ are given by

$$\alpha_r = \begin{cases} \rho_2 & r = 0 \\ \rho_1 \sigma_r & r = 1, 2, 3. \end{cases} \tag{46}$$

Due to the boundary conditions (28′), which hold for all field variables, partial integrations may be performed freely in any integral expression without taking care of the contributions from the boundary.

The second integral in the Hamiltonian is an invariant which could have been omitted without disturbing the invariance of the scheme. Also from the point of view of the derivation of the meson field equations (41) this term remains arbitrary. As will be seen later, the inclusion of this term is, however, necessary in order to avoid the appearance of singular terms in the static interaction of the type of a δ-function.

In terms of the variables in configuration space the source densities (14) are

$$\boldsymbol{M}_\mu = g_1 d \sum_i \boldsymbol{\tau}^{(i)} i\beta^{(i)} \gamma_\mu^{(i)} \delta\left(x - x^{(i)}\right)$$

$$\boldsymbol{S}_{\mu\nu} = \frac{g_2}{2\kappa} d \sum_i \boldsymbol{\tau}^{(i)} i\beta^{(i)} \left[\gamma_\mu^{(i)}, \gamma_\nu^{(i)}\right] \delta\left(x - x^{(i)}\right) \tag{47}$$

with $\delta\left(x - x^{(i)}\right) = \delta\left(x_0 - x_0^{(i)}\right) \delta\left(\vec{x} - \vec{x}^{(i)}\right).$

Eight of these fifteen quantities contain a factor of the same order of magnitude as the ratio between the velocities of the nuclear particles and the velocity of light while the seven other quantities remain finite in the "non-relativistic" approximation. The last mentioned quantities are

$$\left.\begin{aligned} \boldsymbol{N} &= \frac{1}{i} \boldsymbol{M}_4 = g_1 d \sum_i \boldsymbol{\tau}^{(i)} \delta\left(x - x^{(i)}\right) \\ \vec{\boldsymbol{S}} &= (\boldsymbol{S}_{32}, \boldsymbol{S}_{13}, \boldsymbol{S}_{21}) = \frac{g_2}{\kappa} d \sum_i \boldsymbol{\tau}^{(i)} \rho_3^{(i)} \vec{\sigma}^{(i)} \delta\left(x - x^{(i)}\right) \\ \vec{\boldsymbol{P}} &= (\boldsymbol{S}_{01}, \boldsymbol{S}_{02}, \boldsymbol{S}_{03}) = \frac{g_2}{\kappa} d \sum_i \boldsymbol{\tau}^{(i)} \vec{\sigma}^{(i)} \delta\left(x - x^{(i)}\right). \end{aligned}\right\} \tag{48}$$

The expression for the Hamiltonian (45) shows that the energy of free mesons is always positive. The new formalism thus constitutes a consistent scheme satisfying all theoretical requirements.

4. Derivation of the static interaction potential. Electric quadrupole moment of the deuteron.

By the method developed in M.R. it is easy to derive the expressions for the "static" interaction between the nuclear particles. For this purpose, let us consider the equations determining the "static" parts of the meson fields, i. e.

$$\left.\begin{aligned} \boldsymbol{F}_r^0 &= -\frac{\partial \boldsymbol{V}^0}{\partial x_r} \\ \kappa^2 \boldsymbol{V}^0 &= -\frac{\partial \boldsymbol{F}_s^0}{\partial x_s} + \boldsymbol{N} \end{aligned}\right\} \tag{49}$$

$$\left.\begin{aligned} \kappa^2 \boldsymbol{U}_r^0 + \frac{\partial \boldsymbol{G}_{rs}^0}{\partial x_s} &= 0 \\ \boldsymbol{G}_{rs}^0 &= \frac{\partial \boldsymbol{U}_s^0}{\partial x_r} - \frac{\partial \boldsymbol{U}_r^0}{\partial x_s} + \boldsymbol{S}_{rs} \end{aligned}\right\} \tag{50}$$

obtained from (41) and (42) by cancelling the time-derivatives $\dot{\boldsymbol{U}}_r$, $\dot{\boldsymbol{F}}_r$ and the source densities $\boldsymbol{T}_r$ and $\boldsymbol{M}_r$ which are proportional to the velocities of the heavy particles.

From (49) we get

$$\frac{\partial^2 \boldsymbol{V}^0}{\partial x_r \partial x_r} - \kappa^2 \boldsymbol{V}^0 = -\boldsymbol{N}. \tag{51}$$

Using the relation

$$\frac{\partial \boldsymbol{U}_r^0}{\partial x_r} = 0, \tag{52}$$

which follows from the first equation (50), we get further from (50)

$$\frac{\partial^2 \boldsymbol{U}_r^0}{\partial x_s \partial x_s} - \kappa^2 \boldsymbol{U}_r^0 = \frac{\partial \boldsymbol{S}_{rs}}{\partial x_s} \tag{53}$$

r and s running from 0 to 3.

We shall now find a solution $\boldsymbol{V}^0$ of (51) satisfying the boundary conditions (28′). Such a solution may be written as a FOURIER series

$$\left.\begin{aligned} \boldsymbol{V}^0(x) &= \sum_{p_0} \boldsymbol{V}^0_{p_0}(\vec{x})\, e^{\frac{i}{\hbar} p_0 x_0} \\ (x) &= (x_0, \vec{x}) \end{aligned}\right\} \tag{54}$$

where the sum is extended over all values (34) of p_0 and $\boldsymbol{V}^0_{p_0}(\vec{x})$ is a function of $\vec{x} = (x_1, x_2, x_3)$, only.

Expanding $\boldsymbol{N}$ in a similar way

$$\left.\begin{aligned} \boldsymbol{N}(x) &= \sum_{p_0} \boldsymbol{N}_{p_0}(\vec{x})\, e^{\frac{i}{\hbar} p_0 x_0} \\ \boldsymbol{N}_{p_0}(\vec{x}) &= \frac{1}{d} \int_{-\frac{d}{2}}^{\frac{d}{2}} \boldsymbol{N}(x)\, e^{-\frac{i}{\hbar} p_0 x_0}\, dx_0 \end{aligned}\right\} \tag{55}$$

we get by introduction of (54) and (55) into (51) the differential equations

$$\Delta \boldsymbol{V}^0_{p_0} - \left(\kappa^2 + \left(\frac{p_0}{\hbar}\right)^2\right) \boldsymbol{V}^0_{p_0} = -\boldsymbol{N}_{p_0} \tag{56}$$

for the functions $\boldsymbol{V}^0_{p_0}(\vec{x})$.

Equation (56) is of the same type as equation (11) in M.R. Thus, we have the solution

$$\left.\begin{aligned} \boldsymbol{V}^0_{p_0}\left(\vec{x}\right) &= \int \boldsymbol{N}_{p_0}\left(\vec{x}'\right) \frac{e^{-\sqrt{\kappa^2+\left(\frac{p_0}{\hbar}\right)^2}\cdot r}}{4\pi r}\, dv' \\ r &= \left|\vec{x}-\vec{x}'\right| \end{aligned}\right\} \qquad (57)$$

analogous to the equation (14) in M.R.

Hence, with the help of (55), the function $\boldsymbol{V}^0(x)$ in (54) may be written

$$\boldsymbol{V}^0(x) = \frac{1}{d}\int \boldsymbol{N}(x')\,\Phi(x-x')\,d\omega' \qquad (58)$$

where

$$\left.\begin{aligned} \Phi(x-x') &= \sum_{p_0} \frac{e^{-\sqrt{\kappa^2+\left(\frac{p_0}{\hbar}\right)^2}\cdot r}}{4\pi r}\, e^{\frac{i}{\hbar}p_0(x_0-x_0')} \\ &= \varphi(r) + 2\sum_{n=1}^{\infty} \frac{e^{-\kappa r\sqrt{1+\left(\frac{\kappa'}{\kappa}\right)^2 n^2}}}{4\pi r}\cos\left[n\kappa'(x_0-x_0')\right] \end{aligned}\right\} \qquad (59)$$

with

$$\left.\begin{aligned} \varphi(r) &= \frac{e^{-\kappa r}}{4\pi r} \\ \text{and}\quad \kappa' &= \frac{2\pi}{d}. \end{aligned}\right\} \qquad (60)$$

The fact that (58) is the solution of (51) may be expressed by the equation

$$\frac{\partial^2\Phi(x-x')}{\partial x_r\,\partial x_r} - \kappa^2\,\Phi(x-x') = -d\,\delta(x-x') \qquad (61)$$

where

$$\delta(x-x') = \delta(x_0-x_0')\,\delta\left(\vec{x}-\vec{x}'\right).$$

Similarly, we get a solution of (53) of the form

$$\boldsymbol{U}_r^0 = -\frac{1}{d}\int \frac{\partial \boldsymbol{S}_{rs}(x')}{\partial x'_s}\, \Phi(x-x')\, d\omega' \tag{62}$$

an expression which by partial integration is seen to satisfy the condition (52).

We now separate the field variables into a static and a non-static part by the equations*

$$\boldsymbol{F}_r = \boldsymbol{F}_r^0 + \boldsymbol{F}_r^1, \; \boldsymbol{U}_r = \boldsymbol{U}_r^0 + \boldsymbol{U}_r^1. \tag{63}$$

If we introduce these expressions into the Hamiltonian (45) the first integral in $\mathcal{H}$ separates exactly into a "static" and a non-static part, all cross terms vanishing on account of (49) and (50). For the "static part" of the two first integrals in (45) we get

$$\mathcal{H}_F^0 = \frac{1}{2d}\int\left\{(\boldsymbol{F}_r^0)^2 + \kappa^2(\boldsymbol{V}_r^0)^2 + \frac{1}{2}(\boldsymbol{G}_{rs}^0)^2 + \kappa^2(\dot{\boldsymbol{U}}_r^0)^2\right\} d\omega$$
$$- \frac{1}{8d}\int \boldsymbol{S}_{rs}^2\, d\omega.$$

By partial integrations and with the help of (49), (50), (58) and (62) this expression may be written

* As in M. R. it is easy to find a canonical transformation effecting a separation of the type (63). The corresponding unitary operator $\mathcal{S}$ is simply

$$\mathcal{S} = e^{\frac{i}{\hbar c d}\int (\boldsymbol{F}_r^0 \boldsymbol{U}_r - \boldsymbol{U}_r^0 \boldsymbol{F}_r)\, d\omega}$$

$$\mathcal{H}_F^0 = \frac{1}{2d}\int\left\{\boldsymbol{V}^0\boldsymbol{N} + \frac{1}{2}\boldsymbol{G}_{rs}^0\boldsymbol{S}_{rs}\right\}d\omega - \frac{1}{8d}\int\boldsymbol{S}_{rs}^2\,d\omega$$

$$= \frac{1}{2d^2}\int\left\{\boldsymbol{N}(x)\boldsymbol{N}(x')\Phi(x-x') + \frac{1}{2}\big(\boldsymbol{S}_{rs}(x)\boldsymbol{S}_{rt}(x') \right. \quad (64)$$

$$\left. + \boldsymbol{S}_{rt}(x)\boldsymbol{S}_{rs}(x')\big)\frac{\partial^2\Phi(x-x')}{\partial x_s\,\partial x_t}\right\}d\omega\,d\omega' + \frac{1}{8d}\int\boldsymbol{S}_{rs}^2\,d\omega.$$

We have now to introduce the expressions (48) for $\boldsymbol{N}$ and $\boldsymbol{S}_{rs}$. Since we are at the moment interested in the static interaction potential only we may everywhere put $\rho_3 = 1$, $\rho_3 - 1$ being of second order in the velocities of the heavy particles. In this approximation $\vec{\boldsymbol{S}}$ is equal to $\vec{\boldsymbol{P}}$ and

$$\frac{1}{2}\left(\boldsymbol{S}_{rs}(x)\boldsymbol{S}_{rt}(x') + \boldsymbol{S}_{rt}(x)\boldsymbol{S}_{rs}(x')\right) = \vec{\boldsymbol{P}}(x)\vec{\boldsymbol{P}}(x')\,\delta_{st}. \quad (65)$$

Using (65) and (61), we thus get from (64)

$$\mathcal{V}_n = \frac{1}{2d^2}\int\left\{\boldsymbol{N}(x)\boldsymbol{N}(x')\Phi + \vec{\boldsymbol{P}}(x)\vec{\boldsymbol{P}}(x')\frac{\partial^2\Phi}{\partial x_s\,\partial x_s}\right\}d\omega\,d\omega'$$
$$+ \frac{1}{2d}\int\left(\vec{\boldsymbol{P}}\right)^2 d\omega$$

$$= \frac{1}{2d^2}\int\left\{\boldsymbol{N}(x)\boldsymbol{N}(x') + \kappa^2\vec{\boldsymbol{P}}(x)\vec{\boldsymbol{P}}(x')\right\}\Phi\,d\omega\,d\omega'$$

or

$$\mathcal{V}_n = \frac{1}{2}\sum_{i,k}\left(\boldsymbol{\tau}^{(i)}\boldsymbol{\tau}^{(k)}\right)\left[g_1^2 + g_2^2\left(\vec{\sigma}^{(i)}\vec{\sigma}^{(k)}\right)\right]\Phi\left(x^{(i)} - x^{(k)}\right). \quad (66)$$

This expression for the static nuclear potential differs from the corresponding expression in M.R. only by the function $\Phi(x^{(i)} - x^{(k)})$ which here replaces the function

$\varphi = \frac{e^{-\kappa r^{(ik)}}}{4\pi r^{(ik)}}$ in the earlier paper. In the limit of very small d, however, the quantity κ' in (60) and (59) becomes very large and for any finite value of r we may neglect all terms in the expression (59) except the first. In this approximation, the functions Φ and φ are equal and the expression (66) for the static potential is identical with the expression derived in M.R. and does not depend on the variables $x_0^{(i)}$.

The quantities $p_0^{(i)}$ connected with the rest masses of the nuclear particles will then (approximately) be constants of the motion and (in the lowest states) have the value zero.

In the limit of very small d we, thus, get the same results regarding the stationary states of nuclei as in M.R. The same holds for any effect derivable from the theory, the only difference between the two formulations of the meson theory being contained in the relations (13) which reduce the number of undetermined constants from four to two and in this way make the predictions of the new theory more precise. This fact is of importance for the calculation of the quadrupole moment of the deuteron in the ground state. As shown in M.R., the quadrupole moment of the deuteron is due to the occurrence of given non-static directional terms in the expression for the Hamiltonian. If these terms are treated as a small perturbation the perturbed eigenfunction of the ground state will be a superposition of S- and D-states, the coefficients of the D-states being proportional to the matrix element of the perturbation energy corresponding to a transition from the unperturbed S-state to the D-states. The charge-density corresponding to the perturbed eigenfunction of the ground state will then contain cross-terms between the S- and D-states giving rise to a quadrupole moment proportional to the matrix element in question.

The expression for the non-static interaction energy $\mathcal{U}'_n$ of first order in the velocities is obtained by introducing the static fields into the third integral in the Hamiltonian (45). Thus we get, with the help of (49), (58), (62), and by partial integrations

$$\left.\begin{aligned}\mathcal{U}'_n &= -\frac{1}{d}\int\{\boldsymbol{U}_r^0\boldsymbol{M}_r+\boldsymbol{F}_r^0\boldsymbol{T}_r\}\,d\omega\\ &= \frac{1}{d^2}\int\!\!\int\left\{\boldsymbol{N}(x')\,\boldsymbol{T}_r(x)+\boldsymbol{S}_{sr}(x')\,\boldsymbol{M}_s(x)\right\}\frac{\partial\Phi}{\partial x_r}\,d\omega\,d\omega'\end{aligned}\right\}\quad(67)$$

the indices r and s running from 0 to 3.

In the limit of very small d, where Φ is equal to φ and $\frac{\partial\Phi}{\partial x_0}$ is zero, we get by introduction of the expressions (40) and (47) for the source densities

$$\left.\begin{aligned}\mathcal{U}'_n &= \frac{1}{2}\sum_{i,k}{}'\left(\boldsymbol{\tau}^{(i)}\boldsymbol{\tau}^{(k)}\right)\left(\vec{\chi}^{(ik)}\,\mathrm{grad}^{(k)}+\vec{\chi}^{(ki)}\,\mathrm{grad}^{(i)}\right)\varphi\left(r^{(ik)}\right)\\ &\text{with}\\ \vec{\chi}^{(ik)} &= -\frac{g_1g_2}{\varkappa}\left[\rho_2^{(i)}\left(\vec{\sigma}^{(i)}+\vec{\sigma}^{(k)}\right)+\rho_1^{(i)}\rho_3^{(k)}\left(\vec{\sigma}^{(i)}\wedge\vec{\sigma}^{(k)}\right)\right].\end{aligned}\right\}\quad(68)$$

This expression is identical with the formula (85) in M.R. if the undetermined constants occurring there are chosen in accordance with the relations (13) and the foot-note on page 8.

Now, it was shown in part III of the earlier paper that the matrix element of the operator (85) in M. R. corresponding to a transition from a ^3S-state to a ^3D-state is proportional to $f_1 f_2 - g_1 g_2$ to the first order in the velocities. Since the value of the constant f_1 and the signs of all the constants could be chosen arbitrarily in the previous theory the value for the quadrupole moment arising from these terms could be brought into accordance with the empirical result both as regards the sign and magnitude of this quantity. On account of the relations (13), which represent an essential feature of the present theory, it now follows that the matrix element of the operator (68) for any $^3S \rightarrow {}^3D$ transition is zero to the first order in the velocities and, thus, does not give rise to any quadrupole moment of the deuteron in this approximation.

According to the prescription formulated in M.R. (p. 45) we have, therefore, to go one step further in the approximation treatment, i. e. we have in the Hamiltonian (86) in M.R. to retain the terms $-\overset{\odot}{\mathcal{K}}$ and $\mathcal{U}$ which depend linearly on the meson field variables. While the operator $\mathcal{U}_n$ is of the second order in the parameters β and γ $\Big(\beta = \frac{v}{c}\frac{1}{\kappa r}$, $v =$ velocity of the nucleons, $\gamma = \frac{G}{4\pi\hbar c}\frac{1}{\kappa r}$, $G \backsim g_1^2$, g_2^2, or $g_1 g_2\Big)$, these terms will give rise to operators of interaction between the nucleons which are of the third order in the parameters $\frac{v}{c}$, β and γ, and some of these operators will have non-vanishing matrix elements for $^3S - {}^3D$ transitions. Besides the field-dependent interactions $-\overset{\odot}{\mathcal{K}}$ and $\mathcal{U}$, we have also to retain certain direct interactions of the order $\beta^2\gamma$, which have been omitted in the expression for the Hamiltonian (86) in M.R. The most important term of this kind is that which was neglected in (64) by putting $\rho_3 = 1$. In the limit of very small d, this term becomes simply

$$\left.\begin{aligned}\mathcal{U}_n = \frac{1}{2}\left(\frac{g_2}{\kappa}\right)^2 \sum_{i,k}{}' \left(\boldsymbol{\tau}^{(i)}\boldsymbol{\tau}^{(k)}\right)\left(\rho_3^{(i)}\rho_3^{(k)} - 1\right)\left(\vec{\sigma}^{(i)}\,\mathrm{grad}^{(i)}\right)\\ \left(\vec{\sigma}^{(k)}\,\mathrm{grad}^{(k)}\right)\varphi\left(r^{(ik)}\right).\end{aligned}\right\} \quad (69)$$

The result of a calculation of the quadrupole moment due to the mentioned terms in the Hamiltonian can, of course, only be considered reliable if all integrals in this calculation are convergent or, at least, if the result does not depend essentially on the exact value of the "cutting-off" radius. In order to show that this condition is satisfied and that we may expect a quadrupole moment of the deuteron of the right order of magnitude we shall now briefly estimate* the contribution to this quantity arising from the typical term $\mathscr{U}_n$ given by (69). If the two particles of the deuteron are distinguished by the letters N and P and relative coordinates $\vec{x} = \vec{x}^N - \vec{x}^P$ are introduced, the operator (69) may be written

$$\mathscr{U}_n = -\frac{g_2^2}{\kappa^2}\left(\boldsymbol{\tau}^N \boldsymbol{\tau}^P\right)\left(\rho_3^N \rho_3^P - 1\right)\left(\vec{\sigma}^N \operatorname{grad}\right)\left(\vec{\sigma}^P \operatorname{grad}\right)\varphi(r) \qquad (70)$$

where the undefined contributions which correspond to the self-energies have been omitted. Using the representations (110), (111), and (112) in M.R. for the wave functions we get for the matrix element of (70), corresponding to a transition from the (unperturbed) ground state, "0" ($l = 0$, energy E_0) to a ^3D-state with $l = 2$ and energy E

$$(E, l = 2 \,|\, \mathscr{U}_n \,|\, 0) = -\frac{12\, g_2^2}{\kappa^2}\int z^{(2)*}\left(\vec{\sigma}^P \vec{x}_0\right)\left(\vec{\sigma}^N \vec{x}_0\right)\frac{d^2 \varphi}{dr^2} z^{(0)}\, dv$$

where $\vec{x}_0 = \frac{\vec{x}}{r}$ and $z^{(0)}$ and $z^{(2)}$ are given by (113) and (114) in M.R. The introduction of these expressions and integrations over the angles gives

* I am greatly indebted to Mr. I. Nørlund for valuable assistance in these calculations. A complete treatment of the problem is being performed by Dr. L. Hulthén.

$$= 2| \mathcal{U}_n | 0) = -2\sqrt{2}\,\frac{g_2^2}{\kappa^2}\left(\frac{\hbar}{M_0 c}\right)^2 \int_0^\infty \left(\frac{d}{dr}+\frac{2}{r}\right) R_2^{(E)*}\cdot\left(\frac{d}{dr}-\frac{1}{r}\right) R_0 \cdot \frac{d^2\varphi}{dr^2}\, dr \quad (71)$$

where M_0 is the mass of the nuclear particles and R_0 and $R_2^{(E)}$ are the radial wave functions for the unperturbed ground state and a D-state with energy E, respectively. For R_0 we may use the approximate expression (109) in M.R., i. e.

$$\left.\begin{aligned} R_0 &= \sqrt{\frac{\alpha^3\kappa^3}{2}}\, e^{-\frac{\alpha\kappa r}{2}}\, r \\ \text{with} \qquad \alpha &= 2{,}13 \text{ for } \frac{M_m^0}{M_0} = \frac{1}{10}. \end{aligned}\right\} \quad (72)$$

From (72) it follows that

$$\left(\frac{d}{dr}-\frac{1}{r}\right) R_0 = -\frac{\alpha\kappa}{2} R_0$$

and by partial integration (71) may be written

$$, l = 2 | \mathcal{U}_n | 0) = 2\sqrt{2}\,\frac{g_2^2}{4\pi\hbar c}\left(\frac{M_m^0}{M_0}\right)^3 M_0 c^2 \frac{1}{\kappa^2}\cdot (E, l = 2 | \omega | 0)$$

h

$$, l = 2 | \omega | 0) = \frac{4\pi}{\kappa^3}\int_0^\infty R_2^{(E)*}\left[\frac{d^3\varphi}{dr^3} - \frac{1}{r}\frac{d^2\varphi}{dr^2} + \frac{d^2\varphi}{dr^2}\left(\frac{d}{dr}-\frac{1}{r}\right)\right]\left(\frac{d}{dr}-\frac{1}{r}\right) R_0\, dr$$

$$= \frac{4\pi}{\kappa^3}\int_0^\infty R_2^{(E)*}\,\frac{\alpha\kappa}{2}\left[\frac{d^2\varphi}{dr^2}\left(\frac{\alpha\kappa}{2}+\frac{1}{r}\right) - \frac{d^3\varphi}{dr^3}\right] R_0\, dr$$

i. e.

$$\omega = \frac{4\pi}{\kappa^3}\,\frac{\alpha\kappa}{2}\left[\frac{d^2\varphi}{dr^2}\left(\frac{\alpha\kappa}{2}+\frac{1}{r}\right) - \frac{d^3\varphi}{dr^3}\right]. \quad (73)$$

Now by (125) and (126) in M.R. the corresponding quadrupole moment Q is given by the formula

$$\left.\begin{aligned} Q &= \frac{2\sqrt{2}}{5}\int \frac{(0\,|\,r^2\,|\,E, l = 2)\,(E, l = 2\,|\,\mathcal{Q}_n|\,0)}{E_0 - E}\,dE \\ &= -\frac{8}{5}\frac{g_2^2}{4\pi\hbar c}\left(\frac{M_m^0}{M_0}\right)^3 \frac{M_0 c^2}{\kappa^2}\int \frac{(0\,|\,r^2\,|\,E, l = 2)\,(E, l = 2\,|\,\omega\,|\,0)}{|E_0| + E}\,dE. \end{aligned}\right\} \quad (74)$$

On account of the completeness of the eigenfunctions $R_2^{(E)}$ we get a fair estimation of the order of magnitude of Q by writing

$$Q = -\frac{8}{5}\left(\frac{g_2^2}{4\pi\hbar c}\right)\left(\frac{M_m^0}{M_0}\right)^3 \frac{M_0 c^2}{|E_0| + E_m}(0\,|\,r^2\omega\,|\,0)\cdot\frac{1}{\kappa^2} \quad (75)$$

where E_m is the value of E corresponding to the maximum of the numerator of the integrant in (74) and

$$\begin{aligned} (0\,|\,r^2\omega\,|\,0) &= \frac{\alpha^4}{4}\int_0^\infty e^{-(\alpha+1)x}\left[\left(\frac{\alpha}{2}+1\right)x^3 + (\alpha+4)x^2 + (\alpha+8)x + 8\right]dx \\ &= \frac{3\alpha^4}{4}\frac{3\alpha^3 + 12\alpha^2 + 18\alpha + 10}{(\alpha+1)^4} \end{aligned}$$

by help of (72), (73), and (60).

According to (108) and (109) in M.R. we have

$$\frac{g_2^2}{4\pi\hbar c} = 0{,}065, \quad \alpha = 2{,}13, \quad \text{and} \quad \kappa = \frac{M_m^0 c}{\hbar} = \frac{1}{2}10^{13}\,\text{cm}^{-1}$$

for $M_m^0 = \frac{1}{10}M_0$. The value of E_m may be estimated by taking for $R_2^{(E)}$ the Bessel function $\sqrt{kr}\,J_{5/2}(kr)$ $\left(\text{with } k = \frac{1}{\hbar}\sqrt{M_0 E}\right)$. In this way, it was found that the maximum of the numerator in (74) lies at about

$$E_m \sim 0{,}025\,M_0 c^2.$$

Since, further, $E_0 = 0{,}0023\,M_0 c^2$ we get for Q the approximate value

$$Q \sim -3 \cdot 10^{-27}\,\mathrm{cm}^2. \tag{77}$$

Thus, the contribution of the operator $\mathcal{Q}_n$ to the quadrupole moment of the deuteron is of the same order of magnitude as the empirical value[8]) $2{,}73 \cdot 10^{-27}\,\mathrm{cm}^2$, but of opposite sign. All integrals involved in these calculations are convergent, which is due to the circumstance that the matrix elements corresponding to $S-D$ transitions, only, enter into the expression for the quadrupole moment. Although the corresponding corrections to the energy values will depend essentially on the "cutting-off" radius it seems, thus, that the calculation of the quadrupole moment originating from the interactions of third order in the parameters $\frac{v}{c}$, β and γ will lead to an unambiguous result.

In the preceding discussion, we have only considered the limit of very small d in which the space defined by (27) becomes practically identical with the de-Sitter space. The theory is, however, as we have seen, capable of a simple physical interpretation for any value of d, and for all processes depending essentially only on distances larger than the universal distance r_0 (see formula (97) in M.R.) the quantity d may be chosen as large as $2\pi r_0$ without changing the probability for such processes. For a finite value of d the theory implies the existence of particles with different values of the rest mass which perhaps opens the possibility for a unified theory of all known elementary particles with the same spin. In this connection, it should be noticed that the form of the theory is not uniquely determined by the form of the nuclear forces in distances of the order of magnitude κ^{-1}. In fact, we could in the equations of motion (12) and the Lagrangeian (22) replace the source densities $\boldsymbol{M}_\mu$ and $\boldsymbol{S}_{\mu\nu}$ by certain mean values

$\overline{\boldsymbol{M}}_\mu$ and $\overline{\boldsymbol{S}}_{\mu\nu}$ over the variable ρ without spoiling the invariance of the scheme under rotations in $\{R_5\}$.

For instance, we may put

$$\overline{\boldsymbol{M}}_\mu = \int_{P_1}^{P_2} \boldsymbol{M}_\mu \, d\rho \quad \text{and} \quad \overline{\boldsymbol{S}}_{\mu\nu} = \int_{P_1}^{P_2} \boldsymbol{S}_{\mu\nu} \, d\rho$$

where P_1 and P_2 are two points on the same five-dimensional radius vector with ρ equal to $R - \frac{d}{2}$ and $R + \frac{d}{2}$, respectively. As regards phenomena which take place in distances larger than $\frac{d}{2\pi}$ the theory in this form will give the same results as in the form chosen in the present paper, but the probability for a transition between two states corresponding to different values for the rest mass will be zero, at least in the approximation where R is treated as large compared with the dimensions of nuclear physics.

Finally, it should be remarked that the theory developed in this paper of course contains all divergence difficulties inherent in any field theory since we have used the ordinary method of quantization of fields. Consequently, the expression (66) for the static potential, for instance, contains infinite terms corresponding to $i = k$ which have to be discarded.

Appendix.

By introducing suitable space and time coordinates (x, y, z, t) the line element of the de-Sitter world takes the simple form of

$$ds^2 = e^{\frac{2ct}{R}} (dx^2 + dy^2 + dz^2) - c^2 dt^2 \tag{78}$$

where R is a constant. The variable t may be interpreted as the proper time of any observer with "coordinate velocity'' zero, i. e. an observer at a point (x, y, z) = constant. Such observers, who by ROBERTSON[7]) are called equivalent, are in many respects in a similar situation as the observers of a system of inertia in special relativity. The geometry of space is Euclidian and the velocity of light is independent of its direction. The mutual distance of two equivalent observers is, however, not constant in time, the distance measured at time t with a rigid scale of two such observers at the points (x_1, y_1, z_1) and (x_2, y_2, z_2) being $l = re^{\frac{ct}{R}}$ with

$$r = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}.$$

The relative measured velocity is, thus, $\frac{dl}{dt} = \frac{c}{R} l$, i. e. proportional to the distance l.

If we introduce five variables (x_μ) by the equations

$$\left.\begin{aligned} x_1 &= xe^{\frac{ct}{R}},\ x_2 = ye^{\frac{ct}{R}},\ x_3 = ze^{\frac{ct}{R}} \\ x_0 &= R\left(\cosh\frac{ct}{R} - \frac{r^2}{2R^2} e^{\frac{ct}{R}}\right) \\ x_4 &= iR\left(\sinh\frac{ct}{R} + \frac{r^2}{2R^2} e^{\frac{ct}{R}}\right) \\ r &= \sqrt{x^2 + y^2 + z^2} \end{aligned}\right\} \tag{79}$$

(78) may be written

$$ds^2 = \sum_{\mu=0}^{4} dx_\mu^2.$$

Further, we have from (79)

$$\sum_\mu x_\mu^2 = R^2 \tag{80}$$

which shows that the variables (x_μ) may be interpreted as the Cartesian coordinates of a point on a four-dimensional sphere embedded in a Euclidian five-space. The equations (79) may easily be solved with respect to the variables (x, y, z, t). From the last two equations (79) we get

$$x_0 - ix_4 = Re^{\frac{ct}{R}}. \tag{81}$$

Putting for simplicity

$$y_1 = x, \quad y_2 = y, \quad y_3 = z$$

the inverse relations may then be written

$$\left.\begin{aligned} y_r &= \frac{x_r R}{x_0 - ix_4}, \quad (r = 1, 2, 3) \\ t &= \frac{R}{c} ln \frac{x_0 - ix_4}{R}. \end{aligned}\right\} \tag{82}$$

Now, Robertson remarked that a rotation in $\{R_5\}$ defined by (24) generates a transformation of the (y_r, t)-variables which leaves the expression for the line element (78) entirely unchanged, if the transformed variables (y_r', t') are defined by the transformed coordinates (x_μ') by the same relations (79) and (82) as the original variables. The transformation $(y_r, t) \rightarrow (y_r', t')$, thus, connects the space-time coordinates of two sets of equivalent observers in analogy to the Lorentz transformations in the Minkowski world.

A strict accomplishment of Robertson's idea requires, however, a slight modification in the relations (79), (81), and (82). Since t is real we have, according to (81), $x_0 - ix_4 \geq 0$, which shows that the points (x_μ) on the sphere (80) representing the events in the de-Sitter world only cover that hemisphere which lies on the "positive" side of the plane

$$x_0 - ix_4 = 0. \tag{83}$$

This plane is, however, not invariant under all rotations in $\{R_5\}$. The transformed quantity $x_0' - ix_4'$ may, therefore, occasionally be negative, which by (82) would make the corresponding t' complex.

Thus, we have to set up a correspondence between the events (y_r, t) and the points on the "negative" hemisphere, too. This may be done by letting two points (x_μ) and $(-x_\mu)$ on the same diameter of the sphere correspond to the same event (y_r, t). While the equations (79), (81), and (82) are still valid for points on the positive hemisphere, we get then for all points on the sphere (80) the following relations

$$\left.\begin{aligned} y_r &= \frac{x_r R}{x_0 - i x_4} \\ t &= \frac{R}{c} \ln \frac{|x_0 - i x_4|}{R} \end{aligned}\right\} \quad (84)$$

instead of (82).

The general orthogonal transfoımation (24) with positive determinant can be compounded from special rotations of the following types:

I.

$$\left.\begin{aligned} x'_r &= x_r, \qquad (r = 1, 2, 3) \\ x'_0 - i x'_4 &= \lambda (x_0 - i x_4) \\ x'_0 + i x'_4 &= \frac{1}{\lambda} (x_0 + i x_4) \end{aligned}\right\} \quad (85)$$

where λ is an arbitrary real number.

This transformation leaves the expression $\sum_\mu x_\mu^2$ and the equation (83) invariant and has a determinant $\Delta_5 = +1$. By (84) we get for the corresponding transformation in the de-Sitter world

$$\left.\begin{aligned} y'_r &= \frac{y_r}{\lambda}, \qquad (r = 1, 2, 3) \\ t' &= t + \frac{R}{c} \ln |\lambda| \end{aligned}\right\} \quad (86)$$

i. e. a change of time origin with a corresponding change of scale in space. For $\lambda = -1$ (84) reduces to

$$\left.\begin{aligned} y'_r &= -y_r, \ (r = 1, 2, 3) \\ t' &= t \end{aligned}\right\} \quad (87)$$

which represents a spatial reflexion at the origin.

$$\text{II.} \quad \left.\begin{aligned} x'_r &= \sum_{s=1}^{3} a_{rs} x_s + \frac{a_r}{R}(x_0 - ix_4), \ (r = 1, 2, 3) \\ x'_0 - ix'_4 &= x_0 - ix_4 \\ x'_0 + ix'_4 &= x_0 + ix_4 - \frac{2}{R} \sum_{r,s=1}^{3} a_r a_{rs} x_s - \frac{\sum_{r=1}^{3} a_r^2}{R^2}(x_0 - ix_4) \end{aligned}\right\} \quad (8$$

where a_r and a_{rs} are real numbers satisfying the relations

$$\sum_{t=1}^{3} a_{rt} a_{st} = \sum_{t=1}^{3} a_{tr} a_{ts} = \delta_{rs}$$
$$\Delta_3 = |a_{rs}| = +1.$$

This transformation leaves the expressions $\sum_\mu x_\mu^2$ and $x_0 - ix_4$ invariant, it has a determinant $\Delta_5 = +1$ and generates, according to (84), a spatial rotation and translation in the de-Sitter space

$$\left.\begin{aligned} y'_r &= \sum_{s=1}^{3} a_{rs} y_s + a_r, \ (r = 1, 2, 3) \\ t' &= t. \end{aligned}\right\} \quad (89)$$

$$\text{III.} \quad \left.\begin{aligned} x'_0 &= x_0, \quad x'_2 = x_2, \quad x'_3 = x_3 \\ x'_1 &= x_1 \cos\Theta + x_4 \sin\Theta \\ x'_4 &= -x_1 \sin\Theta + x_4 \cos\Theta \\ \operatorname{tg}\Theta &= i\frac{v}{c} \text{ or with } \Theta = i\varepsilon \\ \operatorname{tgh}\varepsilon &= \frac{v}{c}. \end{aligned}\right\} \quad (90)$$

(90) leads, by (84), to the transformation

$$\left.\begin{aligned} x' &= R\coth\frac{\varepsilon}{2}\cdot\frac{2\frac{x}{R}\coth\varepsilon-1-\frac{r^2}{R^2}+e^{-\frac{2ct}{R}}}{\coth^2\frac{\varepsilon}{2}-2\frac{x}{R}\coth\frac{\varepsilon}{2}+\frac{r^2}{R^2}-e^{-\frac{2ct}{R}}} \\ t' &= t+\frac{R}{c}\ln\left\{\sinh^2\frac{\varepsilon}{2}\left|\coth^2\frac{\varepsilon}{2}-2\frac{x}{R}\coth\frac{\varepsilon}{2}+\frac{r^2}{R^2}-e^{-\frac{2ct}{R}}\right|\right\} \\ y'e^{\frac{ct'}{R}} &= ye^{\frac{ct}{R}},\quad z'e^{\frac{ct'}{R}} = ze^{\frac{ct}{R}}, \end{aligned}\right\}\quad (91)$$

which gives the relation between the space and time variables of two freely moving systems of equivalent observers analogous to that afforded by a special LORENTZ transformation. The transversely measured distances are the same for both systems and the motion of the origin $x' = y' = z' = 0$ of one system with respect to the other system is that of a freely moving particle which for $t = 0$ has a coordinate velocity v in the direction of the x-axis. (ROBERTSON, loc. cit.[7])). For events with large negative values of t, the second equation (91) shows clearly the necessity of replacing the equations (82) used by ROBERTSON by the equations (84).

In a small region on the de-SITTER sphere around the point $(R, 0, 0, 0, 0)$ the equations (79) reduce to the equations (25) if we neglect all terms of higher order in $\frac{x}{R}, \frac{y}{R}, \frac{z}{R}, \frac{ct}{R}$ than the first.

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